

ON THE FILTRATION IN HOMOLOGY INDUCED BY THE ATYAH-HIRZEBRUCH SPECTRAL SEQUENCE

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ABSTRACT. In this paper we discuss some relation between the Postnikov tower of a spectrum and the Atiyah-Hirzebruch spectral sequence. We use it to show that the following two filtrations in singular homology agree: the first filtration is given by the Atiyah-Hirzebruch spectral sequence for oriented bordism, and the second one is given by the dimension of the singular part of each cycle. To be more precise we represent each homology class by a map from a stratifold and filter by the dimension of the singular part.

1. INTRODUCTION

In 1946 Steenrod raised the following question [2]:

Given a “nice” space, can every integral homology class be represented by a map from a closed oriented manifold?

This question was answered negatively by Thom. If one replaces manifolds with stratifolds, then the answer is yes. Stratifolds, which were defined by Kreck in [4], are generalization of manifolds, which have also a singular part (we discuss stratifolds in section 5).

We are interested in the following question:

Given a class in integral homology, what is the minimal dimension of the singular part in a stratifold that represents it?

or how “far” it is from being representable.

Our answer (theorem 18) is that the filtration in the integral homology groups $H_p(X, \mathbb{Z})$ given by all classes that can be represented by stratifolds with a singular part of dimension at most $p - r - 2$ agrees with the filtration given by the Atiyah-Hirzebruch spectral sequence (AHSS) for oriented bordism:

$$E_{p,0}^\infty \subseteq \dots E_{p,0}^4 \subseteq E_{p,0}^3 \subseteq E_{p,0}^2 \cong H_p(X, \mathbb{Z}).$$

To prove this we discuss some relation between the Postnikov tower of a spectrum and the AHSS (some connection between those two was studied by Maunder in [8]). We show (theorem 12) that for every homology theory h , CW complex X , and integers p, q , and r , the general term in the AHSS is given by

$$E_{pq}^r = \text{Im} \left(h_{p+q}^{(q+r-2)}(X^p) \rightarrow h_{p+q}^{(q)}(X^{p+r-1}) \right).$$

We define a natural transformation:

$$\Phi : h_n^{(r)}(X) \rightarrow h_{n-1}(X^{n-r-1})$$

that induces the differential in the AHSS. We show that Φ fits into a long exact sequence (theorem 7):

$$\begin{aligned} \dots \rightarrow h_n(X^{n-r-1}) \rightarrow h_n(X) \rightarrow h_n^{(r)}(X) \rightarrow h_{n-1}(X^{n-r-1}) \rightarrow \\ \rightarrow h_{n-1}^{(r)}(X^{n-r-1}) \oplus h_{n-1}(X) \rightarrow h_{n-1}^{(r)}(X) \rightarrow h_{n-2}(X^{n-r-1}) \rightarrow \dots \end{aligned}$$

In the case of oriented bordism (or, more generally, bordism theories) the theories $h_n^{(r)}$ and the natural transformation Φ , have a simple description using stratifolds.

The organization of the paper is as follows:

In section 2 we discuss the basic properties of the Postnikov tower of a homology theory, define the natural transformation Φ and prove the exactness of the sequence mentioned above.

In section 3 we give some applications for this long exact sequence. We show that since the Steenrod problem is not true, there is a space X and an integer n such that

$$Ker(\pi_n(X^n) \rightarrow \Omega_n^{SO}(X^n)) \subsetneq Ker(\pi_n(X^n) \rightarrow H_n(X^n)).$$

In particular this gives a way to find elements in the kernel of the map

$$\Omega_*^{SO}(X) \rightarrow H_*(X \times BSO).$$

In section 4 we describe the general term in the AHSS as mentioned above.

In section 5 we review stratifolds and stratifold homology.

In section 6 we describe the Postnikov tower of a bordism theory using stratifolds and prove our main theorem regarding the filtrations in integral homology.

In the appendix we prove a smooth approximation theorem for maps between stratifolds. A result of this is that every locally finite CW complex is homotopy equivalent to a stratifold having the same set of cells. It also gives a way to control the singular part of a stratifold without changing the class it represents in homology.

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2. THE POSTNIKOV TOWER OF A HOMOLOGY THEORY

Remark. All spaces are assumed to be CW complexes, and for a CW complex X we denote by X^k its k^{th} skeleton.

Let h be a generalized homology theory. One constructs a sequence of homology theories and natural transformations between them, called the Postnikov tower, in the following way:

$$h \rightarrow \dots \rightarrow h^{(2)} \rightarrow h^{(1)} \rightarrow h^{(0)}.$$

The theories $h^{(r)}$ have the property that the map $h_n \rightarrow h_n^{(r)}$ is an isomorphism for $n \leq r$, and $h_n^{(r)}$ is trivial for $n > r$ (h_n stands for $h_n(pt)$, the n^{th} coefficient group). These properties determine $h^{(r)}$ completely. For a proof of existence and uniqueness see [9, Ch II, 4.13 and 4.18].

Example 1. If h is a connective homology theory, that is $h_n = 0$ for $n < 0$, then $h^{(0)}$ is naturally isomorphic to homology with coefficients in h_0 (for CW complexes). If h is oriented bordism, Ω^{SO} , then $h^{(0)}$ is naturally isomorphic to integral homology. Later on we show that in the case of Ω^{SO} , or other bordism theories such as Ω^{Spin} , the theories $h^{(r)}$ can be given a geometric description using stratifolds.

Remark 2. It follows from uniqueness that for $r' \geq r$ we have $(h^{(r')})^r = h^{(r)}$. This implies that in all our constructions and propositions one can replace h by $h^{(r')}$.

By induction, using excision, we get the following:

Lemma 3. *Let X be a CW complex. Then $h_n^{(r)}(X^k)$ is trivial if $k + r < n$.*

Again, using excision and induction, one proves the following:

Lemma 4. $h_{r+k}^{(r)}(X^k, X^{k-1}) \cong h_r \otimes C_k(X)$, where $C_k(X)$ is the k^{th} cellular chain.

This gives a nice description of $h_{r+k}^{(r)}(X^k)$:

Lemma 5. *There is a natural isomorphism $h_{r+k}^{(r)}(X^k) \cong \text{Ker}(h_r \otimes C_k(X) \rightarrow h_r \otimes C_{k-1}(X))$.*

Proof. Look at the exact sequence of the triple (X^k, X^{k-1}, X^{k-2}) :

$$0 \rightarrow h_{r+k}^{(r)}(X^k, X^{k-2}) \rightarrow h_{r+k}^{(r)}(X^k, X^{k-1}) \rightarrow h_{r+k-1}^{(r)}(X^{k-1}, X^{k-2}).$$

We conclude that $h_{r+k}^{(r)}(X^k, X^{k-2}) \cong \text{Ker}(h_r \otimes C_k(X) \rightarrow h_r \otimes C_{k-1}(X))$. The lemma follows from the fact that the map $h_{r+k}^{(r)}(X^k) \rightarrow h_{r+k}^{(r)}(X^k, X^{k-2})$ is an isomorphism. \square

Proposition 6. *Let X be a CW complex, then the map $h_n(X, X^k) \rightarrow h_n^{(r)}(X, X^k)$ is an isomorphism if $n \leq k + r + 1$.*

Proof. We prove it for finite dimensional CW complexes by induction on the dimension. This, together with the fact that for (additive) generalized homology theories we have [7]

$$h_*(X) = \text{colim}(h_*(X^m))$$

will imply the statement for the case where X is infinite dimensional.

The proposition is trivial if $\dim(X) \leq k$, since then both groups vanish. Assume that the statement is true for every Y such that $\dim(Y) = m - 1 \geq k$, in particular for X^{m-1} . Using the long exact sequence for the triple (X^m, X^{m-1}, X^k) we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} h_{n+1}(X^m, X^{m-1}) & \rightarrow & h_n(X^{m-1}, X^k) & \rightarrow & h_n(X^m, X^k) & \rightarrow & h_n(X^m, X^{m-1}) & \rightarrow & h_{n-1}(X^{m-1}, X^k) \\ \downarrow (1) & & \downarrow (2) & & \downarrow (3) & & \downarrow (4) & & \downarrow (5) \\ h_{n+1}^{(r)}(X^m, X^{m-1}) & \rightarrow & h_n^{(r)}(X^{m-1}, X^k) & \rightarrow & h_n^{(r)}(X^m, X^k) & \rightarrow & h_n^{(r)}(X^m, X^{m-1}) & \rightarrow & h_{n-1}^{(r)}(X^{m-1}, X^k). \end{array}$$

The maps (2) and (5) are isomorphisms by our assumption. We have the following commutative diagram, where all horizontal maps are isomorphisms by excision and Mayer-Vietoris:

$$\begin{array}{ccccccc} h_j(X^m, X^{m-1}) & \rightarrow & \oplus h_j(D^m, S^{m-1}) & \rightarrow & \oplus h_j(S^m, *) & \rightarrow & \oplus h_{j-m}(S^0, *) \\ \downarrow (a) & & \downarrow & & \downarrow & & \downarrow (b) \\ h_j^{(r)}(X^m, X^{m-1}) & \rightarrow & \oplus h_j^{(r)}(D^m, S^{m-1}) & \rightarrow & \oplus h_j^{(r)}(S^m, *) & \rightarrow & \oplus h_{j-m}^{(r)}(S^0, *). \end{array}$$

For $j = n$ the map (b) is an isomorphism, since $j - m \leq r$, so the same is true for (a) which is (4). For $j = n + 1$ we either have $\oplus h_{j-m}^{(r)}(S^0, *) = 0$ (if $n = k + r + 1$ and $m = k + 1$), or else (b) is an isomorphism. In any case (b) is surjective and the same is true for (a) which is equal to (1). Now we deduce by the five lemma that (3) is an isomorphism. \square

We use this isomorphism to define a natural transformation

$$\Phi : h_n^{(r)}(X) \rightarrow h_{n-1}(X^{n-r-1})$$

as the composition

$$h_n^{(r)}(X) \rightarrow h_n^{(r)}(X, X^{n-r-1}) \rightarrow h_n(X, X^{n-r-1}) \rightarrow h_{n-1}(X^{n-r-1}).$$

This is better seen using the following diagram:

$$\begin{array}{ccccc} h_n(X) & \rightarrow & h_n(X, X^{n-r-1}) & \rightarrow & h_{n-1}(X^{n-r-1}) \\ \downarrow & & \downarrow \cong & & \downarrow \\ h_n^{(r)}(X) & \rightarrow & h_n^{(r)}(X, X^{n-r-1}) & \rightarrow & h_{n-1}^{(r)}(X^{n-r-1}). \end{array}$$

Proposition 7. *The following sequence is exact:*

$$\begin{aligned} \dots \rightarrow h_n(X^{n-r-1}) \rightarrow h_n(X) \rightarrow h_n^{(r)}(X) \rightarrow h_{n-1}(X^{n-r-1}) \rightarrow \\ \rightarrow h_{n-1}^{(r)}(X^{n-r-1}) \oplus h_{n-1}(X) \rightarrow h_{n-1}^{(r)}(X) \rightarrow h_{n-2}(X^{n-r-1}) \rightarrow \dots \end{aligned}$$

Proof. This can be seen by diagram chasing (for the diagram above), using two facts: the isomorphisms

$$h_k(X, X^{n-r-1}) \rightarrow h_k^{(r)}(X, X^{n-r-1})$$

for all $k \leq n$ (proposition 6) and the fact that $h_n^{(r)}(X^{n-r-1})$ is trivial (lemma 3). \square

Corollary 8. $Im(h_n(X) \rightarrow h_n^{(r)}(X)) \cong Im(h_n(X) \rightarrow h_n(X, X^{n-r-1}))$.

Proof. This follows from the fact that both kernels are equal to

$$Im(h_n(X^{n-r-1}) \rightarrow h_n(X)),$$

using the exact sequence above and the sequence for the pair. \square

3. STEENROD REALIZATION PROBLEM

Steenrod's question, which was mentioned in the introduction, can be rephrased as asking whether the map $\Omega_n^{SO}(X) \rightarrow H_n(X)$ is surjective (here n is arbitrary). Note that this map is the localization map, using the identification $H_n \cong (\Omega_n^{SO})^{(0)}$. Using the exact sequence in proposition 7, this question is equivalent to the question whether the following map is injective:

$$\Omega_{n-1}^{SO}(X^{n-1}) \rightarrow H_{n-1}(X^{n-1}) \oplus \Omega_{n-1}^{SO}(X).$$

By exactness,

$$Ker(\Omega_{n-1}^{SO}(X^{n-1}) \rightarrow \Omega_{n-1}^{SO}(X)) = Im(\Omega_n^{SO}(X, X^{n-1}) \rightarrow \Omega_{n-1}^{SO}(X^{n-1})).$$

By cellular approximation, the map $\Omega_n^{SO}(X^n, X^{n-1}) \rightarrow \Omega_n^{SO}(X, X^{n-1})$ is surjective, so

$$Ker(\Omega_{n-1}^{SO}(X^{n-1}) \rightarrow \Omega_{n-1}^{SO}(X)) = Im(\Omega_n^{SO}(X^n, X^{n-1}) \rightarrow \Omega_{n-1}^{SO}(X^{n-1})).$$

$\Omega_n^{SO}(X^n, X^{n-1})$ is generated by the n -cells, so $Im(\Omega_n^{SO}(X^n, X^{n-1}) \rightarrow \Omega_{n-1}^{SO}(X^{n-1}))$ is generated by the attaching maps.

We deduce that Steenrod's problem (in dimension $n+1$) is equivalent to the following question.

Given a CW complex X , does

$$Ker(\pi_n(X^n) \rightarrow \Omega_n^{SO}(X^n)) = Ker(\pi_n(X^n) \rightarrow H_n(X^n))?$$

Remark 9. The equivalence is in the sense that every element which belongs to the right side but not to the left side corresponds to a non representable class in some X with the given n skeleton.

Let X be a CW complex and $[f : M \rightarrow X]$ an element in $\Omega_*^{SO}(X)$. The classifying map for the stable tangent bundle of M induces a map

$$\Omega_*^{SO}(X) \rightarrow H_*(X \times BSO).$$

This map is known to be a rational isomorphism (see for example [6] 18.51).

Taking X to be a point, this map is injective, i.e. one can detect the cobordism class of M by its image. It would be nice if this was true for every space X . Unfortunately, this is not the case:

Corollary 10. *The map $\Omega_*^{SO}(X) \rightarrow H_*(X \times BSO)$ need not be injective.*

Proof. Since we know that the Steenrod problem is not true, there is a space X and an integer n such that

$$Ker(\pi_n(X^n) \rightarrow \Omega_n^{SO}(X^n)) \subsetneq Ker(\pi_n(X^n) \rightarrow H_n(X^n)),$$

i.e. there is a strict inclusion. Let $[f : S^n \rightarrow X]$ be an element on the right side but not on the left side. Since the tangent bundle of a sphere is stably trivial, the map $S^n \rightarrow X \times BSO$ factors through $X \times *$, hence the image of $[S^n \rightarrow X]$ in $H_*(X \times BSO)$ is zero. \square

4. THE AHSS IN TERMS OF THE POSTNIKOV TOWER

We start by proving the following:

Lemma 11. *Let X be a CW complex. Then the following maps are surjective:*

- 1) $h_n^{(r)}(X) \rightarrow h_n^{(r)}(X, X^{n-r-2})$,
- 2) $h_n(X, X^{n-r-2}) \rightarrow h_n^{(r)}(X, X^{n-r-2})$.

Proof. 1) This follows from the long exact sequence for the pair and lemma 3.

2) Using the isomorphism $h_n(X, X^{n-r-2}) \rightarrow h_n^{(r+1)}(X, X^{n-r-2})$ (proposition 6), it is enough to show that the map $h_n^{(r+1)}(X, X^{n-r-2}) \rightarrow h_n^{(r)}(X, X^{n-r-2})$ is surjective. We look at the following diagram:

$$\begin{array}{ccccc} h_n^{(r+1)}(X, X^{n-r-2}) & \rightarrow & h_n^{(r+1)}(X, X^{n-r-1}) & \rightarrow & h_{n-1}^{(r+1)}(X^{n-r-1}, X^{n-r-2}) \\ \downarrow (1) & & \downarrow (2) & & \downarrow (3) \\ h_n^{(r)}(X, X^{n-r-2}) & \xrightarrow{(4)} & h_n^{(r)}(X, X^{n-r-1}) & \rightarrow & h_{n-1}^{(r)}(X^{n-r-1}, X^{n-r-2}). \end{array}$$

Here (2) is an isomorphism by proposition 6, (3) is an isomorphism by excision, and (4) is injective since $h_n^{(r)}(X^{n-r-1}, X^{n-r-2})$ is trivial. Now the lemma follows by a diagram chase. \square

Theorem 12. *Let X be a CW complex and h a generalized homology theory, then the E_{pq}^r term ($r \geq 2$) in the AHSS is naturally isomorphic to*

$$Im \left(h_{p+q}^{(q+r-2)}(X^p) \rightarrow h_{p+q}^{(q)}(X^{p+r-1}) \right).$$

Proof. We have the following natural isomorphism (see [1] I,7):

$$E_{pq}^r = \frac{Im \left(h_{p+q}(X^p, X^{p-r}) \rightarrow h_{p+q}(X^p, X^{p-1}) \right)}{Im \left(h_{p+q+1}(X^{p+r-1}, X^p) \rightarrow h_{p+q}(X^p, X^{p-1}) \right)}.$$

By the long exact sequence for the triple $(X^{p+r-1}, X^p, X^{p-1})$ we have

$$Im \left(h_{p+q+1}(X^{p+r-1}, X^p) \xrightarrow{\partial} h_{p+q}(X^p, X^{p-1}) \right) \cong Ker \left(h_{p+q}(X^p, X^{p-1}) \rightarrow h_{p+q}(X^{p+r-1}, X^{p-1}) \right).$$

Since ∂ factors through $h_{p+q}(X^p, X^{p-r})$, we get that

$$Ker \left(h_{p+q}(X^p, X^{p-1}) \rightarrow h_{p+q}(X^{p+r-1}, X^{p-1}) \right) \subseteq Im \left(h_{p+q}(X^p, X^{p-r}) \rightarrow h_{p+q}(X^p, X^{p-1}) \right).$$

This implies that $E_{pq}^r \cong Im \left(h_{p+q}(X^p, X^{p-r}) \rightarrow h_{p+q}(X^{p+r-1}, X^{p-1}) \right)$. Look at the following diagram:

$$\begin{array}{ccccc} h_{p+q}(X^p, X^{p-r}) & \rightarrow & h_{p+q}(X^{p+r-1}, X^{p-r}) & \rightarrow & h_{p+q}^{(q)}(X^{p+r-1}, X^{p-r}) \\ \downarrow & & \downarrow & & \downarrow (1) \\ h_{p+q}(X^p, X^{p-1}) & \rightarrow & h_{p+q}(X^{p+r-1}, X^{p-1}) & \xrightarrow{(2)} & h_{p+q}^{(q)}(X^{p+r-1}, X^{p-1}). \end{array}$$

(1) is injective by the long exact sequence for the triple $(X^{p+r-1}, X^{p-1}, X^{p-r})$ and lemma 3;

(2) is an isomorphism by proposition 6.

Denote $f : h_{p+q}(X^p, X^{p-r}) \rightarrow h_{p+q}^{(q)}(X^{p+r-1}, X^{p-r})$. From (1) and (2) it follows that

$$E_{pq}^r \cong Im \left(h_{p+q}(X^p, X^{p-r}) \rightarrow h_{p+q}(X^{p+r-1}, X^{p-1}) \right) \cong Im(f).$$

From lemma 11 it follows that

$$Im(f) = Im \left(h_{p+q}^{(q+r-2)}(X^p) \rightarrow h_{p+q}^{(q)}(X^{p+r-1}, X^{p-r}) \right).$$

Since the map $h_{p+q}^{(q)}(X^{p+r-1}) \rightarrow h_{p+q}^{(q)}(X^{p+r-1}, X^{p-r})$ is injective, we have that

$$Im(f) \cong Im \left(h_{p+q}^{(q+r-2)}(X^p) \rightarrow h_{p+q}^{(q)}(X^{p+r-1}) \right).$$

\square

Remark 13. One can show that the differential in the spectral sequence is induced by the natural transformation

$$\Phi : h_n^{(r)}(X) \rightarrow h_{n-1}(X^{n-r-1}).$$

The E^∞ page.

We look at the case $0 \ll r$. By prop. 6, the map $h_{p+q}(X^p) \rightarrow h_{p+q}^{(q+r-2)}(X^p)$ is an isomorphism for $p+1 < r$. Assume for simplicity that $h_{p+q}^{(q)}(X^{p+r-1}) \rightarrow h_{p+q}^{(q)}(X)$ is an isomorphism for r big enough (this is the case, for example, if we assume that h is connective). In this case we will have $E_{pq}^r = E_{pq}^{r+1}$, so

$$E_{pq}^\infty = \text{Im} \left(h_{p+q}(X^p) \rightarrow h_{p+q}^{(q)}(X) \right).$$

Look at the following diagram:

$$\begin{array}{ccccc} h_{p+q+1}(X, X^p) & \rightarrow & h_{p+q}(X^p) & \rightarrow & h_{p+q}(X) \\ \downarrow (1) & & \downarrow & & \downarrow \\ h_{p+q+1}^{(q)}(X, X^p) & \rightarrow & h_{p+q}^{(q)}(X^p) & \rightarrow & h_{p+q}^{(q)}(X). \end{array}$$

Using the exactness of the rows and the fact that (1) is an isomorphism, we conclude (by diagram chasing) that

$$\text{Ker} \left(h_{p+q}(X^p) \rightarrow h_{p+q}^{(q)}(X) \right) = \text{Ker} (h_{p+q}(X^p) \rightarrow h_{p+q}(X)) + \text{Ker} \left(h_{p+q}(X^p) \rightarrow h_{p+q}^{(q)}(X^p) \right).$$

We know by proposition 7 that

$$\text{Ker} \left(h_{p+q}(X^p) \rightarrow h_{p+q}^{(q)}(X^p) \right) = \text{Im} (h_{p+q}(X^{p-1}) \rightarrow h_{p+q}(X^p)),$$

which implies that:

$$E_{pq}^\infty = \text{Im} (h_{p+q}(X^p) \rightarrow h_{p+q}(X)) / \text{Im} (h_{p+q}(X^{p-1}) \rightarrow h_{p+q}(X)),$$

which is the standard form.

The E^2 page.

The case $r = 2$ has the following form:

$$E_{pq}^2 = \text{Im} \left(h_{p+q}^{(q)}(X^p) \rightarrow h_{p+q}^{(q)}(X^{p+1}) \right).$$

Note that by the long exact sequence for the pair, we have

$$\text{Im} \left(h_{p+q}^{(q)}(X^p) \rightarrow h_{p+q}^{(q)}(X^{p+1}) \right) \cong h_{p+q}^{(q)}(X^p) / \text{Im} \left(h_{p+q+1}^{(q)}(X^{p+1}, X^p) \rightarrow h_{p+q}^{(q)}(X^p) \right),$$

and by lemmas 4 and 5 this is naturally isomorphic to

$$\text{Ker} (h_q \otimes C_p(X) \rightarrow h_q \otimes C_{p-1}(X)) / \text{Im} (h_q \otimes C_{p+1}(X) \rightarrow h_q \otimes C_p(X)),$$

which is by definition $H_p(X, h_q)$, as we know from the standard presentation of the AHSS.

5. STRATIFOLDS AND STRATIFOLD HOMOLOGY

Stratifolds are generalization of manifolds. They were introduced by Kreck [4] and used in order to define a bordism theory, denoted by SH_* , which is naturally isomorphic to singular homology for CW complexes. Kreck also defined a cohomology theory using stratifolds which is defined on the category of smooth oriented manifolds (without boundary but not necessarily compact). It is denoted by SH^* and is naturally isomorphic to singular cohomology.

Stratifolds.

Kreck defined stratifolds as spaces with a certain sheaf of functions, called the smooth functions, fulfilling certain properties, but for our purpose the following definition is enough (these stratifolds are also called p-stratifolds).

Stratifolds are constructed inductively in a similar way to the way we construct CW complexes. We start with a discrete set of points denoted by X^0 and define inductively the set of smooth functions, which in the case of X^0 are all real functions.

Suppose X^{k-1} together with a smooth set of functions is given. Let W be an n -dimensional smooth manifold, “the n strata” with boundary and a collar c , and f a continuous map from the boundary of W to X^{n-1} . We require that f will be proper and smooth, which means that its composition with every smooth map from X^{n-1} is smooth. Define $X^n = X^{n-1} \cup_f W$. The smooth maps on X^n are defined to be those maps $g : X^n \rightarrow \mathbb{R}$ which are smooth when restricted to X^{n-1} and to W and such that for some $0 < \delta$ we have $gc(x, t) = gf(x)$ for all $x \in \partial W$ and $t < \delta$.

Among the examples of stratifolds are manifolds, real algebraic varieties [3], and the one point compactification of a smooth manifold. The cone over a stratifold and the product of two stratifolds are again stratifolds.

We can also define stratifolds with boundary, which are analogous to manifolds with boundary. A main difference is that every stratifold is the boundary of its cone, which is a stratifold with boundary.

Given two stratifolds with boundary (T', S') and (T'', S'') and an isomorphism $f : S' \rightarrow S''$, there is a well defined stratifold structure on the space $T' \cup_f T''$, that is called the gluing. On the other hand, given a smooth map $g : T \rightarrow \mathbb{R}$ such that there is a neighborhood of 0 which consists only of regular values the preimages $g^{-1}((-\infty, 0]) = T'$ and $g^{-1}([0, \infty)) = T''$ are stratifolds with boundary, and T is isomorphic to the gluing $T' \cup_{Id} T''$.

To obtain singular homology we specialize our stratifolds in the following way: we use compact stratifolds, require that their top stratum be oriented and the codimension one stratum will be empty.

Remark. Regarding regularity, a condition that is often required, see [5].

Stratifold homology.

Stratifold homology was defined by Kreck in [4]. We will describe here a variant of this theory called parametrized stratifold homology, which is naturally isomorphic to it for CW complexes. In this paper we will refer to parametrized stratifold homology just as stratifold homology and use the same notation for it.

(parametrized) Stratifold homology is a homology theory, denoted by SH_* . It is naturally isomorphic to integral homology and gives a new geometric point of view on it.

Definition 14. Let X be a topological space and $n \geq 0$, define $SH_n(X)$ to be $\{g : S \rightarrow X\} / \sim$, i.e., bordism classes of maps $g : S \rightarrow X$, where S is a compact oriented stratifold of dimension n and g is a continuous map. We often denote the class $[g : S \rightarrow X]$ by $[S, g]$ or by $[S \rightarrow X]$. $SH_n(X)$ has a natural structure of an Abelian group, where addition is given by disjoint union of maps and the inverse is given by reversing the orientation. If $f : X \rightarrow Y$ is a continuous map, then the induced map $f_* : SH_n(X) \rightarrow SH_n(Y)$ is given by composition.

One constructs a boundary operator and then we get the following:

Theorem 15. (*Mayer-Vietoris*) *The following sequence is exact:*

$$\dots \rightarrow SH_n(U \cap V) \rightarrow SH_n(U) \oplus SH_n(V) \rightarrow SH_n(U \cup V) \xrightarrow{\partial} SH_{n-1}(U \cap V) \rightarrow \dots$$

where the first map is induced by inclusions and the second is the difference of the maps induced by inclusions.

SH_* with the boundary operator is a homology theory.

Theorem 16. *There is a natural isomorphism of homology theories $\Phi : SH_* \rightarrow H_*$.*

Proof. See for example [11]. Φ is given by $\Phi_n([S, f]) = f_*([S])$, where $[S] \in H_n(S, \mathbb{Z})$ is the fundamental class of S . \square

6. APPROXIMATIONS FOR STRATIFOLD HOMOLOGY

Define the following sequence of variants of stratifold homology: $SH_p^{(k)}(X) = [f : S \rightarrow X] / \sim$, where S is as before but has empty strata in codimension $< k + 2$ or, equivalently, its singular part is of dimension at most $p - k - 2$. The same condition must hold for the bordism relation. Note that $SH_p^{(0)}(X) = SH_p(X) \cong H_p(X, \mathbb{Z})$. Clearly, there are the following natural transformations:

$$\Omega_p^{SO}(X) \rightarrow \dots SH_p^{(2)}(X) \rightarrow SH_p^{(1)}(X) \rightarrow SH_p^{(0)}(X) \cong H_p(X, \mathbb{Z}).$$

The map $\Omega_n^{SO} \rightarrow SH_n^{(r)}$ is an isomorphism for $n \leq r$ and $SH_n^{(r)}$ is trivial for $n > r$. This proves the following:

Proposition 17. $(\Omega^{SO})^{(r)} \cong SH^{(r)}$.

We use this geometric description of $(\Omega^{SO})^{(r)}$ in order to describe the AHSS for Ω^{SO} :

$$E_{pq}^r = \text{Im} \left(SH_{p+q}^{(q+r-2)}(X^p) \rightarrow SH_{p+q}^{(q)}(X^{p+r-1}) \right).$$

When $q = 0$, we have that

$$E_{p0}^r = \text{Im} \left(SH_p^{(r-2)}(X^p) \rightarrow SH_p(X^{p+r-1}) \right).$$

When $r \geq 2$, this is equal to

$$E_{p0}^r = \text{Im} \left(SH_p^{(r-2)}(X) \rightarrow SH_p(X) \right).$$

In other words we proved:

Theorem 18. *For a CW complex X , the filtration in singular homology given by the AHSS*

$$E_{p,0}^\infty \subseteq \dots E_{p,0}^4 \subseteq E_{p,0}^3 \subseteq E_{p,0}^2 \cong H_p(X, \mathbb{Z}),$$

agrees with the filtration given by all classes in homology that are represented by maps from stratifolds with singular part of dimension at most $p - r - 2$.

Remark 19. We will prove later (corollary 24) that both filtration agree with the filtration induced by the images of

$$\Omega_p(X) \rightarrow \dots \Omega_p(X, X^{p-4}) \rightarrow \Omega_p(X, X^{p-3}) \rightarrow \Omega_p(X, X^{p-2}) \rightarrow H_p(X, X^{p-2}) \cong H_p(X, \mathbb{Z}).$$

The differential in this description is easily described. For an element α in

$$E_{pq}^r = \text{Im} \left(SH_{p+q}^{(q+r-2)}(X^p) \rightarrow SH_{p+q}^{(q)}(X^{p+r-1}) \right),$$

$d(\alpha)$ lies in

$$E_{p-r, q+r-1}^r = \text{Im} \left(SH_{p+q-1}^{(q+2r-3)}(X^{p-r}) \rightarrow SH_{p+q-1}^{(q+r-1)}(X^{p-1}) \right).$$

Take $[S, f] \in SH_{p+q}^{(q+r-2)}(X^p)$ that is mapped to α .

We claim that $d(\alpha)$ is given by the attaching map of the top stratum of S after using cellular approximation to map it to X^{p-r} . This means, that if S is given by attaching a $p+q$ manifold with boundary $(W, \partial W)$ to some singular part S' , then $[\partial W \rightarrow X^{p-r}]$ represents $d(\alpha)$. This is easily seen from the isomorphism

$$SH_{p+q}^{(q+r-2)}(X^p) \cong \Omega_{p+q}(X^p, X^{p-r})$$

given by forgetting the singular part after some cellular approximation. Then the differential is clearly given by restriction to the boundary.

APPENDIX - SMOOTH APPROXIMATION OF MAPS BETWEEN STRATIFOLDS

In the initial draft of this paper we were interested mainly in proving theorem 18. This was done in a geometric way by explicitly constructing the stratifold that represents each homology class. The proof here is more general, we avoid the technicalities and have some other results. Since some of the techniques used in the original draft are interesting on their own we present them here. The main tool is the approximation theorem.

Lemma 20. *Every continuous map $f : M \rightarrow S$, where M is a manifold with boundary ∂M , S is a stratifold, and f is smooth when restricted to ∂M , is homotopic to a smooth map rel. boundary.*

Proof. We prove it by induction on the dimension of S . This is clear if S is 0-dimensional, since then S is discrete, so f is locally constant. Assume that we know that for stratifolds of dimension $< n$, and let $f : M \rightarrow S$ be a continuous map, where S is of dimension n .

By using the collar of ∂M , we can assume that f is smooth on a neighbourhood of ∂M .

By construction, S is obtained from Σ , the singular part, and a manifold N of dimension n with a boundary ∂N and an (open) collar. Denote by U the singular part together with the collar, and by V the interior of N , that is $N \setminus \partial N$. This is an open cover of S , denote $f^{-1}(U) = U'$ and $f^{-1}(V) = V'$, then this is an open cover of M . We can choose a smooth function $g : M \rightarrow \mathbb{R}$ such that $g|_{U' \setminus V'} = 0$ and $g|_{V' \cup U'} = 1$, and a regular value both of g and of $g|_{\partial M}$, say 0.5. Denote by P its preimage. Then P is a manifold with a boundary, denoted by ∂P . By a standard approximation argument one can show that f is homotopic rel. boundary to a map $\tilde{f} : M \rightarrow S$ with the following properties:

- 1) \tilde{f} is smooth in a neighbourhood of $\partial M \cup g^{-1}([0.5, 1])$;
- 2) $g^{-1}([0.5, 1])$ is mapped into V ;
- 3) $g^{-1}([0, 0.5])$ is mapped into U .

We are left with smoothing $g^{-1}([0, 0.5])$. We can find a manifold M' of dimension n with boundary $\partial M'$, embedded in $g^{-1}([0, 0.5])$ such that \tilde{f} is smooth outside

of M' and in a neighbourhood of $\partial M'$. M' is mapped to U , which is smoothly homotopy equivalent to Σ . This implies that it is homotopic to a new map, which is smooth outside of M' and in a neighbourhood of its boundary, and that the image of M' is contained in Σ . Now we use the inductive step to smooth this map using the fact that Σ is of dimension $< n$. \square

Proposition 21. *Every continuous map $f : S \rightarrow S'$, where S and S' are stratifolds is homotopic to a smooth map.*

Proof. This is proved by induction on the dimension of S using the lemma above. \square

Proposition 22. *Any locally finite, finite dimensional CW complex is homotopy equivalent to a stratifold.*

Proof. This is proved by induction on the dimension using the fact that the attaching maps can be made smooth using lemma 20. \square

Lemma 23. *Let X^d be a CW complex of dimension d and M^m a closed oriented manifold of dimension $m > d$ together with a map $f : M^m \rightarrow X^d$. The element (M^m, f) represents the zero element in $SH_m(X^d)$, and there exists a null bordism, which has singular part of dimension $\leq d$.*

Proof. The first assertion is clear since $SH_m(X^d) = 0$. First assume that X^d is a compact stratifold and f is smooth. In this case the null bordism can be taken to be the mapping cylinder $Cyl(f)$ whose boundary is M^m and whose singular part is X^d . The general case follows from proposition 22 since the image of f is contained in some finite subcomplex that is homotopy equivalent to a stratifold. \square

Corollary 24. *The filtrations given by the images*

$$\Omega_p(X) \rightarrow \dots SH_p^{(2)}(X) \rightarrow SH_p^{(1)}(X) \rightarrow SH_p^{(0)}(X) \cong H_p(X, \mathbb{Z})$$

and

$$\Omega_p(X) \rightarrow \dots \Omega_p(X, X^{p-4}) \rightarrow \Omega_p(X, X^{p-3}) \rightarrow \Omega_p(X, X^{p-2}) \rightarrow H_p(X, X^{p-2}) \cong H_p(X, \mathbb{Z})$$

are equal.

Proof. Given $[S, f] \in SH_p^{(k)}(X)$ we can assume by cellular approximation that S^{p-k-2} , the $p-k-2$ skeleton of S , is mapped to the X^{p-k-2} , the $p-k-2$ skeleton of X . Denote by $(N^p, \partial N^p)$ the manifold we used to get the top stratum of S . Then there is an induced map $(N^p, \partial N^p) \xrightarrow{f} (X, X^{p-k-2})$. By the definition of the bordism relation, $[(S, \emptyset), f] = [(N^p, \partial N^p), f] \in SH_p(X, X^{p-k-2})$.

Let $[(N^p, \partial N^p), f] \in \Omega_p(X, X^{p-k-2})$ be any element. By lemma 23, $(\partial N^p, f|_{\partial N^p})$ represents the zero element in $SH_m(X^{p-k-2})$, and there exists a null bordism, $(S, \partial N^p)$, which has singular part of dimension $\leq p-k-2$. Gluing $(N^p, \partial N^p)$ and $(S, \partial N^p)$ we get a stratifold $N^p \cup_{\partial N^p} S$ of dimension p with a singular part of dimension $\leq p-k-2$ mapped to X , hence an element in $SH_p^{(k)}(X)$, such that its image in $SH_p(X, X^{p-k-2})$ equals to the image of $[(N^p, \partial N^p), f]$. \square

Lemma 23 and its proof have another nice

Corollary 25. *Let X be a CW complex having cells only in even dimensions, e.g. a complex variety. Every homology class can be represented by a map from a stratifold having only even dimensional strata. This can be restated in the following way: let SH_*^{even} be the bordism theory of stratifolds with strata only in even codimensions. Then the map $SH_*^{even}(X) \rightarrow SH_*(X)$ is surjective.*

Proof. In this case the odd dimensional homology groups are trivial, so we may only look at even dimensional ones. Given $\alpha \in SH_{2k}(X)$, we represent it using a stratifold $[S, f]$, and like we did before, we can replace the singular part of S with the $2k - 2$ skeleton of X without changing the homology class. \square

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